Grounding Bohmian Mechanics in Weak Values and Bayesianism

New Journal of Physics 9, 165 (2007)

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Outline

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1. WHY CONSIDER HIDDEN VARIABLES.

Operations versus Explanations

Orthodox Quantum Theory (OQT) is an *Operational Theory*. That is, for the following temporally-ordered *macroscropic* events:

- Preparation procedure c
- Measurement procedure a (that can be *freely chosen* by Alice)
- Measurement outcome A

the theory gives you $P(A|a,c)$. It offers no *explanation* or *interpretation*.

Any additional variables λ are operationally superfluous and so can be defined to be *Hidden Variables*.

Any *model* or *mechanism* which offers some extra *explanation* using HVs is a HV interpretation.

Hidden Variables Interpretations

A HV interpretation (HVI) consists of

1. The set Λ of values of λ.
2. A mapping from c to a probability meausure $d\mu_c(\lambda)$ on Λ.
3. A probability distribution $P(A|a,c,\lambda)$ satisfying

$$\int_\Lambda d\mu_c(\lambda) P(A|a,c,\lambda) = P(A|a,c).$$

In (non-trivial) HVIs, $P(A|a,c,\lambda) \neq P(A|a,c)$.

Why consider Hidden Variables?

1. To *explain* the probabilities that appear in the operational theory.

2. To *explain* the existence of people who perform preparations, choose measurements, and observe results. That is, to explain the things that are *assumed* in the operational theory.

3. Perhaps to suggest research towards a theory that might supersede quantum theory.

2. THE PROBLEM WITH HIDDEN VARIABLES.

Violation of Locality

Now consider two distant parties, with space-like separated measurements and results.

Bell (1964) showed that Quantum Phenomena violate local causality. That is, there does not exist any explanation [Λ, $d\mu_c(\lambda)$, $P(A,B|a,b,c,\lambda)$] of OQT:

$$P(A,B|a,b,c) = \int_\Lambda d\mu_c(\lambda) P(A,B|a,b,c,\lambda)$$

such that

$$P(A|a,B,b,c,\lambda) = P(A|a,c,\lambda).$$

That is, there are some Quantum Phenomena that cannot result from local causes. The trivial case $\lambda = \rho_c$ is no exception.

F$\star\star\star$ Locality

(apologies to Lucien)

The only way to avoid the violation of local causality is to be strictly operational.[1]

However this does *not* mean that OQT respects local causality.

Rather, being a strict operationalist means refusing to consider explanations, and so refusing to admit the concept of local causality.

So one could argue (Bell certainly did) that nonlocality is not a *problem* of HV models, but rather a *feature* of OQT revealed by considering HV models.

[1] Or to deny the reality of the experience of distant observers, or to deny free will, or perhaps to allow retrocausation.

Nonuniqueness is a real problem

There are infinitely many *nonlocal* HVIs compatible with experience. See Bacciagaluppi and Dickson, Found. Phys. (1999) and Gambetta and Wiseman, Found. Phys. (2004) for an even more general formulation.

We could just accept this and say no more. However, if we identify a unique HVI preferred *on physical grounds*, then

1. This would aid pedagogy.

2. This could aid intuition into Quantum Phenomena.

3. This might point towards a theory *beyond* QT.

3. BOHMIAN MECHANICS.

[1]

[1] de Broglie (1926); Bohm (1953) and many others since.

Single-particle Bohmian mechanics

Consider scalar particles for simplicity, and for the moment just a single particle with state $|\psi\rangle$. Then the Bohmian HV is the particle's position $\mathbf{x}$, and

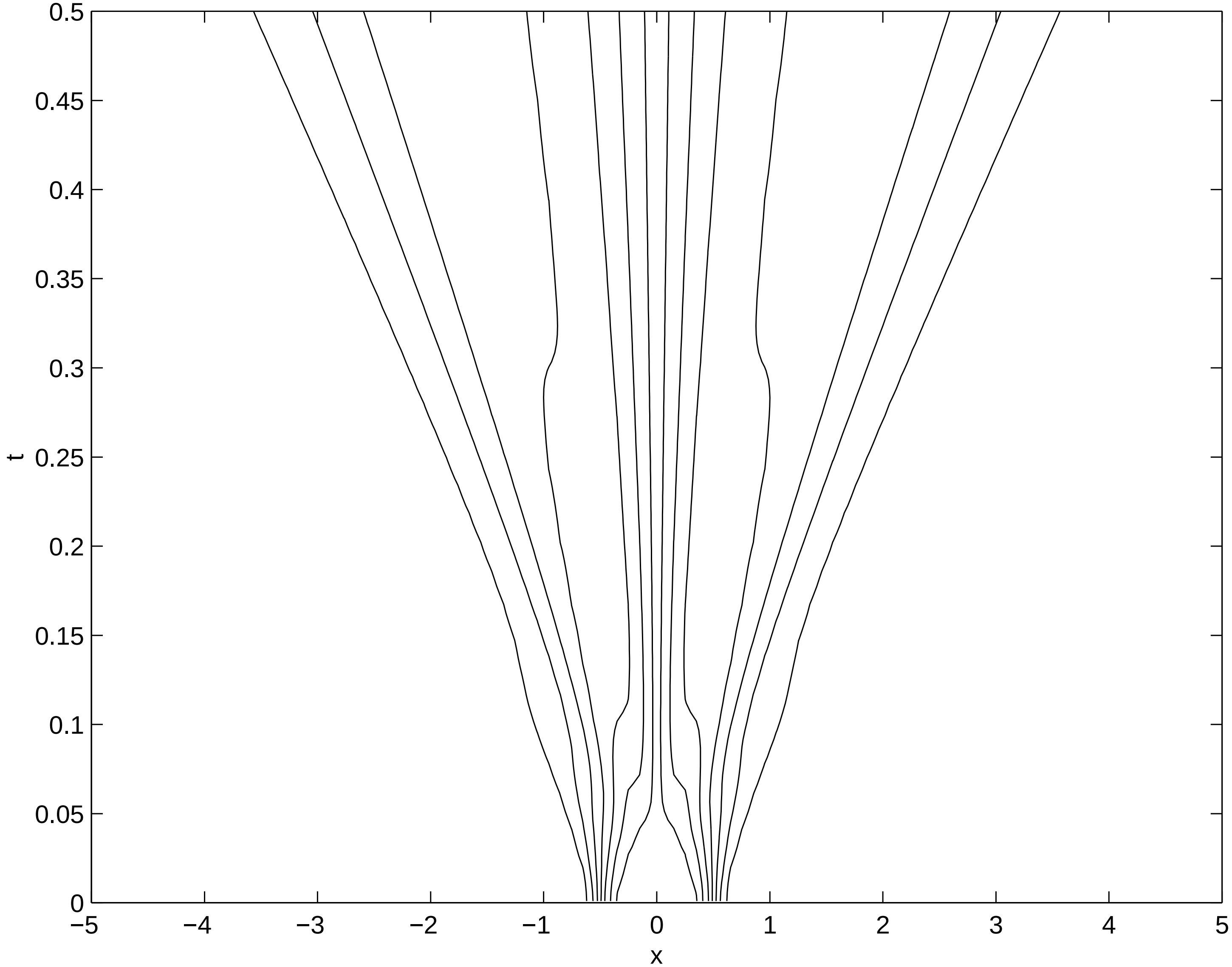
$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{v}(\mathbf{x};t) \equiv \mathbf{j}(\mathbf{x};t)/P(\mathbf{x};t), \\ P(\mathbf{x};t) &= \langle\psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\psi(t)\rangle, \\ \mathbf{j}(\mathbf{x};t) &= (\hbar/m)\mathrm{Im}\,\langle\psi(t)|\mathbf{x}\rangle\nabla\langle\mathbf{x}|\psi(t)\rangle. \end{aligned}$$

This $\mathbf{j}(\mathbf{x};t)$ is the standard *probability current* (flux), which satsifies

$$\frac{\partial}{\partial t}P(\mathbf{x};t) + \nabla\cdot\mathbf{j}(\mathbf{x};t) = 0.$$

This guarantees that if the probability distribution for $\mathbf{x}$ at time t_0 is $P(\mathbf{x};t_0)$ then at time t it will be $P(\mathbf{x};t)$.

An example of Bohmian trjaectories



General Bohmian Mechanics

In general Bohmian mechanics, $\mathbf{x}$ is an ∞-vector including the 3-positions of all the particles and also the values of all the quantized gauge fields at every point in space. It obeys

$$\dot{x}_n = v_n(\mathbf{x};t) = \mathrm{Re}\frac{\langle\Psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|i[\hat{H},\hat{x}_n]|\Psi(t)\rangle}{\hbar\langle\Psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\Psi(t)\rangle}.$$

Here $|\Psi\rangle$ is a *universal* wavefunction or guiding function, not the state of some subsystem (as in OQT).

BM is *nonlocal* because $\dot{x}_n$ depends on *all* the co-ordinates in $\mathbf{x}$.

Bell (1980): "It is a merit of the de Broglie-Bohm version to bring this [nonlocality] out so explicitly that it cannot be ignored."

OQT *emerges* from Bohmian Mechanics

Quantum states for subsystems (as in OQT) *emerge* from BM. Say the universe comprised only an observer o and a system s, and o could assign a pure state to s, then that state would be

$$|\psi_s\rangle \propto \langle \mathbf{x}_o | \Psi \rangle.$$

Unlike OQT, BM defines the observer unambiguously, being made of particles and fields with a definite configuration $\mathbf{x}_o$, which is known (to some approximation) to the observer by introspection.

In addition to the *operational* state $|\psi_s\rangle$ (Hardy, 2004), the system is also characterized by an (unknown) $\mathbf{x}_s$, guided by $|\psi_s\rangle$, to which the observer will assign the distribution

$$\langle \psi_s | \mathbf{x}_s \rangle \langle \mathbf{x}_s | \psi_s \rangle.$$

Aside: Epistemic States Versus Operational States, and Excess Baggage

Note that $|\psi_s\rangle$ is *not* an *epistemic* state for $\mathbf{x}_s$ (unlike the case in Rob's toy theory, where the epistemic states and operational states are identical, both distinct from ontic states.)

Hardy (2004) has proven an *ontological excess baggage theorem* for QM: the *number* of distinct epistemic states (which must be at least as large as the *number* of distinct operational states) is *infinitely* greater than the *dimensionality* of the space of operational states.

In the context of BM it can perhaps be argued that this is related to nonlocality: the operational states are determined by the ontic states of the rest of the universe, which is much bigger than the system.

4. WEAK VALUES AND BOHMIAN DYNAMICS

The problem with j

There are infinitely many expressions for $\mathbf{j}$ that obey

$$\frac{\partial}{\partial t}P(\mathbf{x};t)+\nabla\cdot\mathbf{j}(\mathbf{x};t)=0,$$

while still satisfying "all possible physically meaningful requirements one can put forward for them" (Deotto and Ghirardi, 1998).

Since the "standard" $\mathbf{j}(\mathbf{x})$ has been around since 1926 one might think it would have an operational definition, but it seems not.

The problem is it relates to the velocity of the particle at a particular position $\mathbf{x}$ — quantities that cannot be simultaneously measured.

To solve the problem, turn to *Weak Values* (Aharanov, Albert & Vaidman, 1988) which have a proud history of providing the best operational definition of concepts that orthodox QM cannot define.

Weak Measurements and Weak Values

A precise (or strong) measurement of some observable $\hat{a}$ in general greatly disturbs the quantum state, projecting it into $|A\rangle$.

But if the measurement is *imprecise*, the disturbance *can be* small.

A *weak* measurement of $\hat{a}$ is one which is arbitrarily imprecise, and the disturbance arbitrarily small, such as defined by the following POM in the limit $\sigma \gg a_{\rm max} - a_{\rm min}$:

$$\hat{F}_\sigma(A)dA = (2\pi\sigma^2)^{-1/2}\exp[-(\hat{a}-A)^2/2\sigma^2]dA.$$

A *weak value* is just the *mean* value of a weak measurement.

Simply considering a prepared state $|\psi\rangle$ gives a boring mean value:

$$\left\langle a_{\rm weak}\right\rangle_{|\psi\rangle} = \left\langle a_{\rm strong}\right\rangle_{|\psi\rangle} = \langle\psi|\hat{a}|\psi\rangle.$$

Interesting Weak Values

To be interesting requires *post-selection* [AAV (1988)]. That is, the average of the weak measurement results A is calculated from the sub-ensemble where a *later* strong measurement yields the result corresponding to the state $|\phi\rangle$.

The post-selected weak value can be shown to be given by the simple formula

$$_{\langle\phi|}\left\langle \hat{a}_{\mathrm{w}} \right\rangle_{|\psi\rangle} = \mathrm{Re}\frac{\langle\phi|\hat{a}|\psi\rangle}{\langle\phi|\psi\rangle}.$$

The weak value can lie *outside* the range of eigenvalues of $\hat{a}$ [AAV (1988)], as first verified experimentally [Ritchie, Story & Hulet (1991)].

(This of course cannot happen for a strong measurement of $\hat{a}$.)

Weak-valued v(x)

For a classical ensemble of particles, the (drift) velocity at a position $\mathbf{x}$ could be measured by the measuring the velocity, and post-selecting on measuring the position to be $\mathbf{x}$.

In the quantum case, a strong measurement of the velocity operator $\hat{\mathbf{v}} = i[\hat{H}, \hat{\mathbf{x}}]$ would greatly disturb the particle's position.

Thus I propose the most natural operational definition of $\mathbf{v}(\mathbf{x})$ is:

$$\mathbf{v}(\mathbf{x};t) = {}_{\langle \mathbf{x}|} \left\langle \hat{\mathbf{v}}_{\mathrm{w}} \right\rangle_{|\psi(t)\rangle}.$$

This can be shown to be equivalent to:

$$\mathbf{v}(\mathbf{x};t) \equiv \lim_{\tau \to 0} \tau^{-1} \mathrm{E}[\mathbf{x}_{\mathrm{strong}}(t+\tau) - \mathbf{x}_{\mathrm{weak}}(t) | \mathbf{x}_{\mathrm{strong}}(t+\tau) = \mathbf{x}].$$

Evaluating the “Naively Observable” v(x)

This evaluates to exactly the standard Bohmian expression:

$$\mathbf{v}(\mathbf{x};t) = \mathrm{Re}\frac{\langle\Psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|i[\hat{H},\hat{\mathbf{x}}]|\Psi(t)\rangle}{\hbar\langle\Psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\Psi(t)\rangle} \; !$$

Note that this “works” as a velocity field only because $\hat{H}$ is at most quadratic in the variables conjugate to the HV (that is $\hat{\mathbf{p}}$).

Thus, a naive experimentalist, knowing only that it is necessary to use imprecise measurements in order to avoid altering the system, would, with a large enough ensemble, reconstruct the possible paths of Bohmian particles directly from experimental data.

Note that it is *not* possible to follow a single particle along its trajectory, only to determine the possible trajectories the particles may follow from an identically prepared source.

Determinism: a necessary assumption

Strictly, a naive experimentalist would recognize $\mathbf{v}(\mathbf{x};t)$ only as the *mean* velocity in configuration space — the noise in the weak measurement could be masking variations in the velocity between individual systems that have the same Bohmian position $\mathbf{x}$ at time t.

There are in fact other interpretations (e.g. Nelson, 1966) in which $\mathbf{x}$ is the HV, but in which the motion of $\mathbf{x}$ is stochastic, and $\mathbf{v}(\mathbf{x};t)$ *is* only the mean velocity.

Thus to derive BM from the measured $\mathbf{v}(\mathbf{x};t)$ it is necessary to make the assumption of *determinism*.

5. WEAK VALUES AND BOHMIAN KINEMATICS

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[1]Since HV dynamics are first-order in time, the kinematics is the HV itself, i.e. $\mathbf{x}$ in BM.

Configuration as the HV: an *unnecessary* assumption

It would seem necessary to assume that the HV is the configuration $\mathbf{x}$, as any set of commuting operators can give an HV theory.

For determinism, we need a continuous spectrum (Bub, 1997). But this still allows for momentum, for example (Brown and Hiley, 2001).

However, if we assume the weak-valued velocity, then we can *rule out* replacing $\hat{\mathbf{x}}$ by $\hat{\mathbf{p}}$. This is because $\hat{H}$ is *not* at most quadratic in the variables conjugate to the $\hat{\mathbf{p}}$ (that is, $\hat{\mathbf{x}}$). For example: the Coulomb potential $\propto |\mathbf{x}_j - \mathbf{x}_k|^{-1}$; the cubic Hamiltonian of the gluon field.

That is, a naive experimentalist could determine that $\mathbf{p}$ as the HV has a $P(\mathbf{p};t)$ which is *not* compatible with the naively determined $\mathbf{v}(\mathbf{p};t)$. In general, the *kinematics* $\mathbf{x}$ is singled out by the *dynamics*.

6. PROBABILITY AND BOHMIAN MECHANICS

The probability problem in Bohmian Mechanics

Bohmian mechanics reproduces all of OQT given the kinematics $\mathbf{x}$, the dyanamics

$$v_n(\mathbf{x};t) = \mathrm{Re}\frac{\langle\Psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|i[\hat{H},\hat{x}_n]|\Psi(t)\rangle}{\hbar\langle\Psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\Psi(t)\rangle},$$

and the probability assignment

$$P(\mathbf{x};t_0) = \langle\Psi(t_0)|\mathbf{x}\rangle\langle\mathbf{x}|\Psi(t_0)\rangle.$$

But why should $|\Psi\rangle$ play this dual role?

A deeper question: What is probability?

The radical Bayesian (de Finetti) answer: *Probability is not real*.

$P(\mathbf{x};t_0)$ is only an expression of one observer's beliefs about $\mathbf{x}$. It is known as the prior probability distribution, or *prior*.

How do the objective probabilities of OQT arise?

(Jaynes') Principle of Indifference

"If the statement of a statistical problem is invariant under some transformation, then choose a prior that respects this indifference."

Recall that the problem is specified by the (unkown) $\mathbf{x}(t_0)$ and the (known) $|\Psi(t_0)\rangle$. But there is no particular significance to the time t_0. Therefore the prior should be covariant with respect to translation in time. That is,

$$\frac{\partial}{\partial t} P_{\rm prior}(\mathbf{x};t) = \sum_n \frac{\partial}{\partial x_n}[P_{\rm prior}(\mathbf{x};t)\dot{x}_n(\mathbf{x};t)].$$

If we require that $P_{\rm prior}(\mathbf{x};t) \propto$ function of $\langle \mathbf{x}|\Psi(t_0)\rangle$ and its derivatives, then (Sheldon Goldstein & Ward Struyve, 2007) the *unique* solution is

$$P_{\rm prior}(\mathbf{x};t_0) = \langle \Psi(t_0)|\mathbf{x}\rangle\langle \mathbf{x}|\Psi(t_0)\rangle.$$

Prior and Posterior Distributions

Remember the simple example of a universe comprised only o and s, with an *operational* state for the system of $|\psi_s\rangle \propto \langle \mathbf{x}_o|\Psi\rangle$.

Here it is as if the observer knows her own configuration $\mathbf{x}_0$. Such a degree of self-knowledge is neither realistic nor required.

Nevertheless, because the observer is part of the universe in BM, her knowledge of $\mathbf{x}$ is certainly *not* limited to the prior distribution:

$$P(\mathbf{x};t) \neq \langle \Psi(t)|\mathbf{x}\rangle\langle \mathbf{x}|\Psi(t)\rangle,$$

where $\mathbf{x}$ incorporates $\mathbf{x}_o$. The right-hand-side is what a totally innocent observer believes. The left-hand-side is the *posterior* distribution.

Epistemology, Ontology, and Nomology

As soon as an innocent observer opens her eyes she collapses her state of belief about $\mathbf{x}$ from $P_{\rm prior}(\mathbf{x};t)$ to a much sharper $P(\mathbf{x};t)$, conditioned on her observing the location of macroscopic objects.

This "collapse" is classical/epistemic/psychological. The configuration $\mathbf{x}$ does not suddenly change, and neither does $|\Psi(t)\rangle$[1].

In BM one should not think of $|\Psi(t)\rangle$ as a quantum state but rather a guiding function, the essential constituent of the law of motion.

$P(\mathbf{x};t)$ is epistemic. $\mathbf{x}(t)$ is ontic. $|\Psi(t)\rangle$ is nomic.

[1]This Bayesian updating by an observer in BM is thus similar to the pruning of other branches by an observer in each branch of Everett's universal wavefunction. The difference is that in BM there is a unique real branch singled out by $\mathbf{x}$, and probabilities can be interpreted in the usual way.

7. SOME REMAINING PROBLEMS / DIRECTIONS

Relativistic Invariance

Note: As Ward Struyve discussed, we can use Dirac's original particle-hole formulation, *which is equivalent to standard relativistic QFT* for fermions. Then the fermion positions can be HVs, since

$$\hat{H} = \sum_n c\underline{\hat{\beta}}_n \cdot [\underline{\hat{p}}_n + e\underline{A}(\underline{\hat{q}}_n)] + mc^2\hat{\alpha}_n \tag{1}$$

is at most quadratic in $\hat{p}_n$ (in fact, it is at most linear in $\hat{p}_n$).

Nevertheless, BM, like all HVMs, requires a preferred foliation of space-time, but (of course) it does not allow us to determine what it is. This is unsatisfying.

Perhaps when we have a correct quantum theory of space-time this will be resolved. That is, perhaps it will be found necessary to introduce a preferred foliation.

The nomic versus the epistemic

If $|\Psi\rangle$ (nomic) and $|\psi_s\rangle \propto \langle \mathbf{x}_o|\Psi\rangle$ (epistemic) are so different in nature, how come they are both described by a vector in Hilbert space which evolves unitarily (for an isolated system)?

Perhaps a TOE will specify a unique $|\Psi\rangle$, and this will allow the law of motion to be reformulated so as to removes this apparent similarity.

For example, if $|\Psi\rangle$ is the (assumed unique) solution to $\hat{H}|\Psi\rangle = E_0|\Psi\rangle$, then we can re-express the Bohmian law of motion as:

$$\dot{\mathbf{x}} = \mathrm{Im}\frac{\mathrm{Tr}[\delta(\hat{\mathbf{q}} - \mathbf{x})\,\hat{H}\,\hat{\mathbf{q}}\,\delta(\hat{H} - E_0)]}{\mathrm{Tr}[\delta(\hat{\mathbf{q}} - \mathbf{x})\,\delta(\hat{H} - E_0)]}.$$

Or, for example, it might be possible to show that almost every $|\Psi\rangle$ is compatible with our experience. [In marked contrast to $|\psi_s\rangle$].

Explorations of Theory Space

If we consider *some* guiding function $f(\mathbf{x})$ and *some* law of motion $\dot{\mathbf{x}} = \mathbf{v}(f(\mathbf{x}), \nabla f(\mathbf{x}), \cdots)$, what further restrictions are required to derive (or rule out)

- A type of locality ($\sim$ signal-locality), subsystems, complex structures.
- Intrinsic Unpredictability: a fundamental distinction between epistemic and ontic states.
- Well-motivated priors despite intrinsic unpredictability.
- Concept of an *operational* state (as distinct from epistemic or ontic).
- Violation of Local Causality.
- The singling-out of the 'true' kinematics and dynamics from operational considerations (naive experimentalists etc.)

SUMMARY

1. The probability current in configuration space has a natural operational definition using weak measurements.

2. This operational definition agrees with the standard expression for the quantum probability current.

3. Thus the possible trajectories of the hidden variable $\mathbf{x}$ in the Bohmian interpretation can be determined by a naive experimentalist assuming only that this interpretation is deterministic.

4. Adopting the naively observable velocity of a hidden variable in general, the *asymmetry* between the configuration and the conjugate momenta in physical Hamiltonians singles out the former.

   That is, if the trajectories are to be compatible with the experimentally observable evolution of the probability distribution, the HV must be the configuration $\mathbf{x}$ as in Bohmian mechanics.

5. Given the Bohmian guidance equation for $\mathbf{x}$, the usual quantum distribution for $\mathbf{x}$ can be derived in the context of Bayesian probability theory as the unique prior covariant under translation of the initial time, in accord with Jaynes' principle of indifference.

6. Many interesting open questions.